Degree-two inequalities,clique facets and biperfect graphs

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by

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ABSTRACT

We consider zero-one solutions to inequality systems having inequalities involving only two variables. Special cases include the independent node problem, the node covering problem, and the set packing problem. We develop a unified framework using a class of bidirected, transitively closed graphs and show properties of the zero-one solutions in terms of these graphs. A type of clique inequality is defined and shown to yield facets of the convex hull of zero-one solutions, extending well-known results for the independent node problem. For the latter problem, the development of perfect graphs has been important with regard to characterizing polytopes which have only zero-one vertices. We place our work in that setting, give some results, and conjecture some properties of perfect graphs involving our more general graph structures.

RESUME

Dans cet article, nous considérons un système d'inégalités linéaires en variables 0-1, dans lequel il y a exactement deux coefficients non nuls par inégalité. Cette classe contient comme cas particuliers les problèmes suivants :

- trouver un ensemble stable maximum dans un graphe
- trouver un ensemble minimum de sommets recouvrant un graphe
- problèmes du type "set packing"
- ainsi que quelques problèmes de flot.

Nous proposons ici un traitement unifié de tous ces cas qui utilise une classe de graphes bi-orientés et transitivement fermés, et nous montrons des propriétés des solutions 0-1 en termes de ces graphes. Nous généralisons aux graphes considérés ici,le concept de clique, nous introduisons une inégalité associée et nous montrons qu'elle définit une facette de l'enveloppe convexe des solutions 0-1. Ceci est une généralisation naturelle d'un résultat bien connu pour les graphes non orientés. Concernant ces derniers, les études sur les graphes parfaits ont été importantes pour caractériser des polyèdres n'ayant que des sommets 0-1. Nous proposons ici une généralisation aux graphes bi-orientés du concept de graphe parfait ; nous en montrons quelques propriétés et nous posons des conjectures concernant les graphes bi-parfaits considérés ici.

1. Introduction

We consider inequality systems with zero-one variables where the inequalities have only two variables per inequality. Without loss of generality, there are four possible inequalities for any pair x_i, x_j of variables:

$$\begin{aligned} x_i + x_j &\geq 1; \\ x_i - x_j &\geq 0; \\ -x_i + x_j &\geq 0; \\ -x_i - x_j &\geq -1. \end{aligned}$$

Our main object of study is the convex hull of zero-one solutions to inequality systems made up of inequalities of this type.

Besides being interesting as inequality systems in their own right, as we consider them here, such inequalities arise as logical implications of general zero-one programming problems and have been called degree-two constraints [1, 7, 8, 12]. For example, Guignard and Spielberg [7] build up logical inequalities of this and more general types and exploit them in limiting and guiding the enumerative search in solving zero-one problems. Their propogation procedure is used to fix variables or to finally give what we call a transitively closed graph. Hammer and Ngyen [8]used degree-two inequalities as a key tool in their APOSS procedure.

We will represent these systems of inequalities by a certain type of graph, called bigraphs, which are similiar to the bi-directed graphs for the general matching problem [4].

A case of special interest is the independent node problem. In that problem, a zero-one variable x_i is associated with node i, for each node i, and the inequality $x_i + x_j \leq 1$ is required if the edge $[i,j]$ is present. The nodes i having $x_i = 1$ and satisfying these inequalities form an independent set of nodes, i.e. a set of nodes such that no edge connects any two nodes in the set. Any set packing problem [11] can formally be converted to an independent node problem on the "intersection graph", i.e. the graph with a node i for the $i^{\underline{th}}$ subset and an edge $[i,j]$ if the $i^{\underline{th}}$ and $j^{\underline{th}}$ subset have a nonempty intersection.

The node covering problem involves constraints $x_i + x_j \geq 1$ for every edge $[i,j]$.

The nodes with $x_i = 1$ form a node cover, i.e. every edge meets at least one such node. The complement of a node cover is an independent set, and clearly a node cover containing the minimum number of nodes among all node covers for a given graph is the complement of an independent set containing the maximum number of nodes among all independent sets of the given graph.

Although a general set packing problem can be converted to a node packing problem via the intersection graph, the same is not true of node covering. In the case of set packing, there are many families of sets which will give the same intersection graph. The intersection graph, and its associated cliques to be discussed in section 3, gives a cannoniced representation of any set packing problem.

2. Degree Two Inequalities and Bidirected Graphs

A bidirected graph $G = (N,E)$ is a set N of <u>nodes</u> and a set E of <u>edges</u> where each edge $e \in E$ has two <u>ends</u>. Each <u>end</u> of an edge <u>meets</u> a node $i \in N$ and has an associated sign. The signs for the two ends of an edge need not agree. Thus, an edge e is denoted as the unordered pair $[i,j]$ where i and j are the two nodes met by the ends of edge e. In addition, the signs of the ends of the edge lead to three types of edges:

(+,+) edges $e = [i,j]$ with two plus ends,
(−,−) edges $e = [i,j]$ with two minus ends,
(+,−) edges $e = [i,j]$ with a plus end at i and a minus end at j.

A (−,+)edge of a bidirected graph G is defined accordingly. Note that edges of the form $[i,i]$ may present in the graph. We call such edges <u>loops</u>.

This definition is the same as in Edmonds and Johnson [4]. Here, however, we have a different use of such graphs. A system of inequalities each of which has two zero-one variables can be represented by a bidirected graph having a node i for each variable x_i and an edge for each inequality:

$x_i + x_j \geq 1$ gives a (+,+) edge $[i,j]$;
$-x_i - x_j \geq -1$ gives a (−,−) edge $[i,j]$;
$x_i - x_j \geq 0$ gives a (+,−) edge $[i,j]$.

Thus, for any inequality system with inequalities of these types there is a unique bidirected graph, and any bidirected graph represents such an inequality system.

A loop [i,i] of a (+,+) type corresponds to the inequality $x_i + x_i \geq 1$. In 0-1 variables, this inequality implies $x_i=1$. Similarly, a (-,-) loop gives $-x_i - x_i \geq -1$, or $x_i = 0$. A (+,-) loop does not give an inequality since it would give $x_i - x_i \geq 0$.

Two edges e and e' may meet the same pair of nodes [i,j] if they are of different type, e.g. if e is a (+,+) edge and e' is a (+,-) edge. That is, an inequality system may well include the two distinct inequalities:

$x_i + x_j \geq 1$, and
$x_i - x_j \geq 0$..

Note that the last two inequalities imply a (+,+) loop at node i.

Given a bidirected graph G, we form the <u>transitive closure</u> G^* of G, inductively, by adjoining edges to G as follows:
if node j has two distinct incident edges e and e' with a plus end of e meeting j and a minus end of e' meeting j , then the new edge, if not already in G , to be adjoined to G has as its two ends the other end of e and the other end of e'. To be more explicit, suppose the edge e has other end meeting node i and edge e' has other end meeting node k. Then, the new edge e^* has one end meeting node i, with the same sign as the end of e meeting i, and the other end of e meets node k with the same sign as the end of e' meeting k. It is allowed that i be equal to k or that i be equal to j or that j be equal to k, but edges e and e' are required to be distinct.

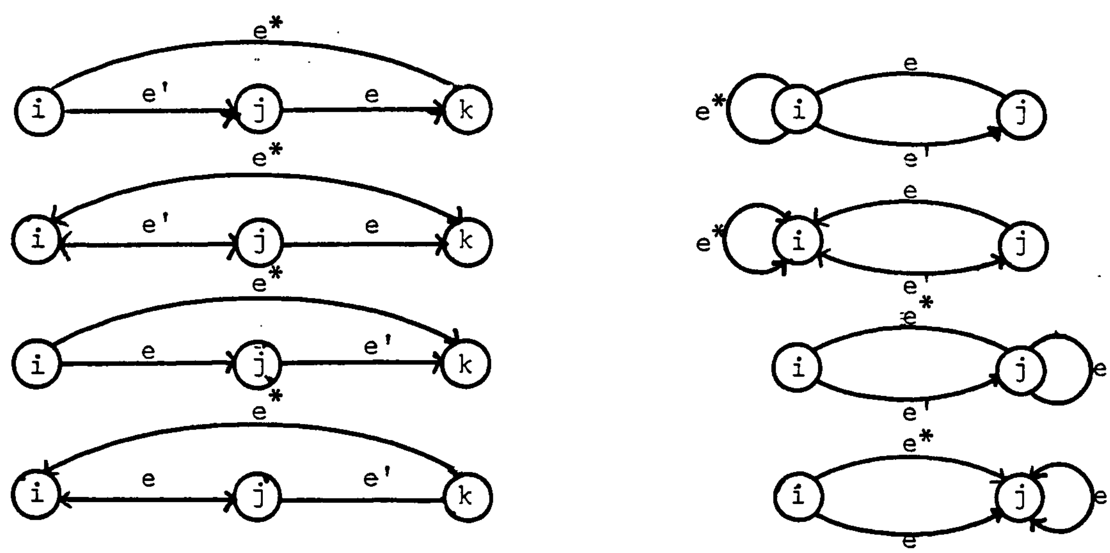

Figure 1

We need not adjoin an edge already present with the same sign pattern because that would correspond to writing the same inequality twice. Similarly, a (+,-) loop can be dropped because it corresponds to the trivial inequality $x_i - x_i \geq 0$.

A transitively closed graph is a bidirected graph G such that G is equal to its transitive closure G^*. That is, all of the edges e^* which might be required to be adjoined to G are already present in G.

A simple bidirected graph is a bidirected graph with no loops and no two edges meeting the same pair of nodes.

Subsequently, we work with simple, transitively closed graphs. For a given graph, the transitive closure may not be simple even if the original graph was. However, a graph which is not simple can always be reduced to a smaller graph as explained below.

First, if a node i has a (+,+) loop, then the inequality $x_i + x_i \geq 1$. must be satisfied so any 0-1 solution must have $x_i = 1$. Likewise, if a (-,-) loop is at node i, then $-x_i - x_i \geq -1$, so $x_i = 0$ is satisfied by every 0-1 solution. In this case, the node i and all incident edges can be deleted from the graph. Clearly, if a node i has both a (+,+) loop and a (-,-) loop then there is no 0-1 solution to the problem. In the next section, we prove the converse: there is a 0-1 solution whenever the transitive closure has no node with both a (+,+) loop and a (-,-) loop.

We also show that if the transitive closure has no loops at all, then for any node i there is a 0-1 solution with $x_i = 0$ and another 0-1 solution with $x_i = 1$.

Second, if there is a (+,-) edge [i,j] and a (-,+) edge [i,j], then $x_i \geq x_j$ and $x_j \geq x_i$ so $x_i = x_j$ and eliminate one variable. In the graph, we "shrink" the two nodes [i,j] together as in figure 2(i). Likewise,

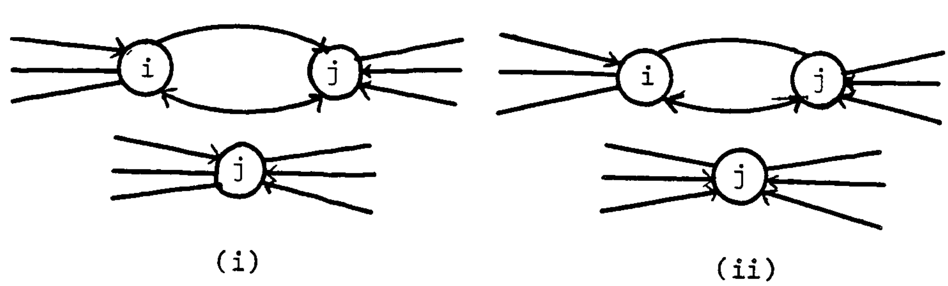

Figure 2

if there is a (+,+) edge [i,j] and a (-,-) edge [i,j] , then $x_i + x_j \geq 1$ and $-x_i - x_j \geq -1$ hold and thus $x_i = 1 - x_j$. We can eliminate x_i using the substitution $x_i \equiv 1 - x_j$. Figure 2(ii) illustrates this case.

We are interested in these graphs because we want to represent inequality systems. Thus, we can consider that a duplicate (+,+) edge [i,j] can be eliminated in the same way that an inequality $x_i + x_j \geq 1$ can be eliminated if it has already appeared in the inequality system. In the same way, changing a variable x_i to $x_i' = 1 - x_i$ does not essentially change the inequality system. This change is represented in the graph by changing all the signs of ends of edges at a given node. That is, changing x_i to $1-x_i$ changes each plus end meeting node i to a minus end and each minus end meeting node i to a plus end without affecting the transitive closure of the bidirected graph.

Therefore, we can say that our graphs are not essentially changed by duplicating an edge, including the signs of its ends, or by changing all of the signs at a node.

The construction of the transitive closure of a given bidirected graph G can be done by "scanning" each edge of the transitive closure G^* exactly once. We first state what the main step, scanning an edge, consists of.

<u>Scan edge e = [i,j]</u> : If e has a plus end at node i, look at every other edge e' with a minus end at i and adjoin e^* , as required in the definition of transitive closure, if it is not already present. If e has a minus end at node i , then look at every other edge e' with a plus end at i. Do the same for node j .

We can start with a list of edges in any order and start scanning each edge in turn. New edges e can be adjoined to the bottom of the list. Terminate when every edge, either originally present or adjoined, has been scanned.

To show that this simple scanning procedure produces the transitive closure requires only showing that for the graph produced, every pair of edges e and e' meeting node i with oppositely signed ends eventually gets looked at. But scanning the edge e or e' furthest down on the list must result in looking at the pair because at that point the other edge is present.

Since the order of work in the scanning step can be kept down to order $n = |N|$ and the number of edges in G^* is at most order n^2 , the total work required to get the transitive closure of a bidirected graph is at most order n^3.

3. Bigraphs

For this section we concern ourselves with simple, tansitively closed bidirected graphs. Call such graphs bigraphs. A 0 - 1 solution to the inequality system associated with a bigraph G is called a 0 - 1 solution for G .

Proposition 1 *The subgraph* G_S *induced by a subset* S *of the nodes of a bigraph* G *is a bigraph.*

Proof. Clearly G_S is bidirected and simple. G_S is transitively closed because every edge of the transitive closure of G_S is an edge of the transitive closure of G . Since G is a bigraph the proposition follows.

The following two constructions are used frequently in both this section and the next.

Construction 0 (Assigning a variable the value 0): For any bigraph G let T be the subset of nodes of G consisting of a node i and all of its neighboring nodes j met by an edge [i,j] having a plus end at node i and let $R = N - T$. For $k \in T$ we construct a partial solution x^T as follows:

$$x_k = \begin{cases} 0 & \text{if } k = i \text{ or } [i,k] \text{ is a } (+,-) \text{ edge.} \\ 1 & \text{if } [i,k] \text{ is a } (+,+) \text{ edge.} \end{cases}$$

Construction 1 (Assigning a variable the value 1): For any bigraph G let U be the subset of nodes of G consisting of a node i and all of its neighboring nodes j met by an edge [i,j] having a minus-end at node i and let $S = N - U$. For $k \in U$ we construct a partial solution x^U as follows:

$$x_k = \begin{cases} 1 & \text{if } k = i \text{ or } [i,k] \text{ is a } (-,+) \text{ edge.} \\ 0 & \text{if } [i,k] \text{ is a } (-,-) \text{ edge.} \end{cases}$$

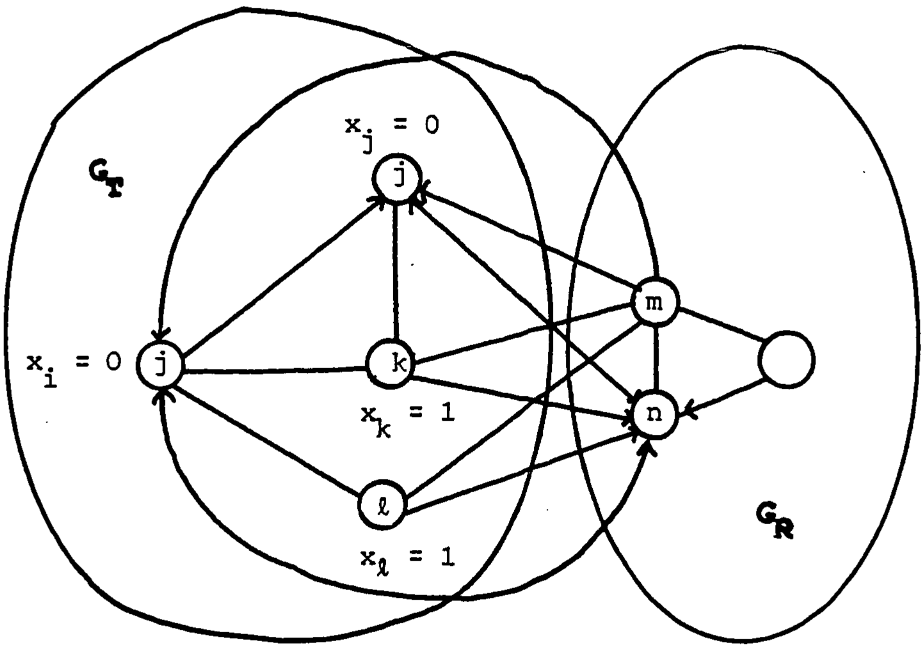

Figure 3: Construction 0

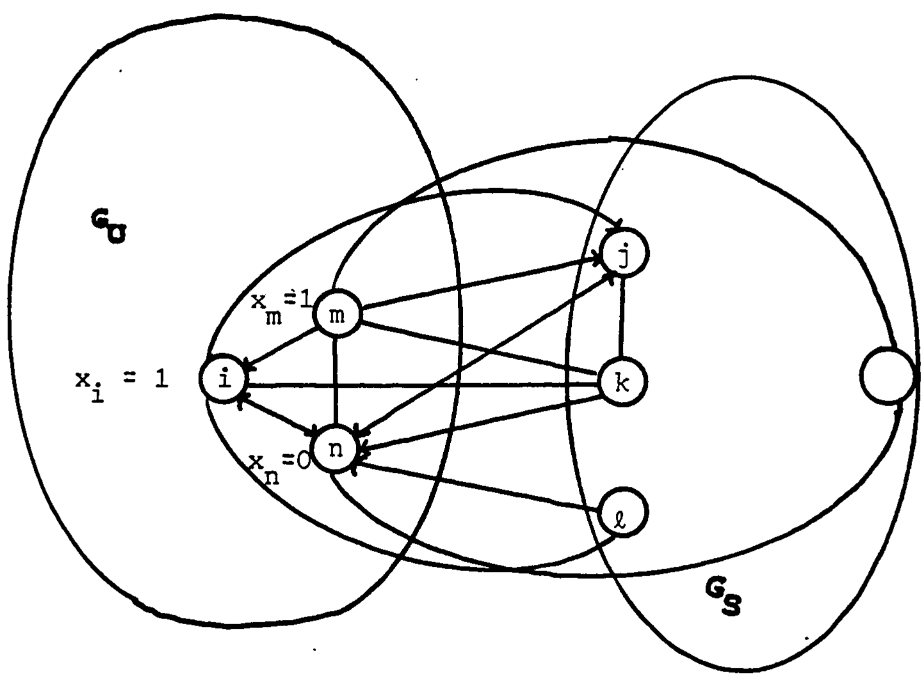

Figure 4: Construction 1

Proposition 2 *Let G be a bigraph and let $i \in N$, T, R, S and U be defined as in Construction 0 and Construction 1, respectively.*

(i) If x^R is a 0-1 solution for the subgraph G_R, then $x = (x^T, x^R)$ is a 0-1 solution for G.

(ii) If x^S is a 0-1 solution for the subgraph G_S, then $x = (x^U, x^S)$ is a 0-1 solution for G.

Proof. We prove part (i) of the proposition; the proof of part (ii) is similar. By construction x^T is a 0-1 solution for G_T. Hence it suffices to show that the inequalities corresponding to edges $e = [j,k]$ with $j \in T$ and $k \in R$ are satisfied by x^T and, more precisely, that they are rendered superfluous by the choice of x^T. We consider the four cases that are possible and show that only two of them can occur in the bigraph G:

(i) $x_j = 0$ and edge e has a plus end at j;
(ii) $x_j = 1$ and edge e has a plus end at j;
(iii) $x_j = 0$ and edge e has a minus end at j;
(iv) $x_j = 1$ and edge e has a minus end at j.

In case (ii) the corresponding inequality is either $x_j - x_k \geqq 0$ or $x_j + x_k \geqq 1$. With x_j fixed to the value 1, the inequality is satisfied and reduces to $x_k \leqq 1$ or $x_k \geqq 0$. Thus in case (ii) the assertion follows. In case (iii) the assertion follows by an analogous argument.
Suppose now case (i) occurs. Node j cannot be equal to the node i chosen in Construction 0 because otherwise $k \in T$ holds. Since $j \in T$, $j \neq i$ and $x_j = 0$ hold, the edge $[i,j]$ is a (+,-) edge. Since the edge $e = [j,k]$ has a plus end at j and since G is transitively closed, it follows that the edge $e^* = [i,k]$ is in G and that it has a plus end at node i. Hence, $k \in T$ follows, contradicting $k \in R = N - T$. Thus case (i) cannot occur. By analogous reasoning, case (vi) cannot occur. Hence, we are either in case (ii) or (iii) and taking any 0-1 solution x^R for G_R yields a 0-1 solution for G when combined with x^T. Proposition 2 follows.

We remark that this proposition is true only because G is transitively closed, which means intuitively, that the implications of setting $x_i = 0$ (or 1) have already been propogated through the graph, see [7].

Proposition 2 does not assure that there is a 0-1 solution for G because there may be no 0-1 solution for G_R. However, the existence of a 0-1 solution for a bigraph G is easily proven from proposition 2 by induction. In fact, more is true.

Proposition 3 *Let G be a bigraph and $k \in N$ be any node of G. Then there exist 0-1 solutions x^0 and x' for G such that $x_j^0 = x_j'$ for all $j \neq k$, $x_k^0 = 0$ and $x_k' = 1$ hold.*

Proof. Let W be the set of nodes of G which are not connected to node k by an edge of G. We apply e.g. construction 0 to any node $i \in W$ and continue to do so until we are left with a subgraph G' with node set N' such that $k \in N'$ and every node of G' is linked to the node k by some edge of G'. If $i \in N'$, $i \neq k$, is linked to node k by an edge having a plus end at node i we set $x_i = 1$ and use Construction 1. It follows that node k is in the set S that results and thus remains unfixed. If $i \in N'$ $i \neq k$, is linked to node k by an edge having a minus end at node i we set $x_i = 0$ and use Construction 0. It follows that node k is in the set R that results and thus remains unfixed. In either case we can apply Construction 0 or 1 again and repeat until node k is an isolated node. Thus x_k can be assigned arbitrarily the value 0 or 1 and proposition 3 follows.

Proposition 4 *Let G be a bigraph and let $i \neq j$ be any two nodes of G. For any assignment of 0-1 values to x_i and x_j which is not excluded by an edge of G (if present) there exits a 0-1 solution for G.*

Proof. Consider first the case where there is no edge $[i,j]$ in G. Then, we prove the proposition by first applying construction 0 or 1, as required, for node i and following with construction 0 or 1, again as required, for node j, which will be in G_R or G_S, as the case may be. Since there is no edge $[i,j]$, construction 0 (or 1) for node i will leave node j in G_R (or G_S) so that either construction can then be applied to node j.

To prove the proposition, in general, requires considering several cases. We consider the case of a $(+,+)$ edge $e = [i,j]$ and leave the other cases to the reader. The values excluded by a $(+,+)$ edge for $[i,j]$ are $x_i = 0$ and $x_j = 0$ since the corresponding inequality is $x_i + x_j \geq 1$.

Thus, we need to show that all three of $x_i = 0$ and $x_j = 1$, $x_i = 1$ and $x_j = 0$, $x_i = 1$ and $x_j = 1$ can occur in 0-1 solutions for G. If construction 0 is applied to node i, then a 0-1 solution for G having $x_i = 0$ and $x_j = 1$ will be produced. Applying construction 1 to node i leaves node j in G_S so that either $x_j = 0$ or $x_j = 1$ is possible in a 0-1 solution for G_S. Using proposition 2, there are 0-1 solutions for G with $x_i = 1$ and x_j equal to either 0 or 1, completing the proof.

Proposition 4 shows that the transitive closure of G gives all of the possible implications on pairs of variables taking on 0-1 values. This result is even stronger than saying that there are no new inequalities on pairs of variables which can be derived as non-negative combinations of the existing inequalities.

Theorem 5 *The convex hull of 0-1 solutions for a bigraph G is a full-dimensioned polytope in $\mathbb{R}^n$.*

The proof of Theorem 5 follows directly from Proposition 3.

4. Clique Facets

A clique in a graph is a maximal, completely connected subgraph of the graph. Thus, the set S of nodes of the clique has the property that for any pair of nodes in S, there is an edge of G meeting that pair of nodes; and adjoining another node of G to S would cause that completely connected property to no longer hold.

Let G be a bigraph with node set N and let G_S be a completely connected subgraph of G with node set S. Define $S_+ \subseteq S$ ($S_- \subseteq S$, respectively) the subset of nodes of G_S which are met by an edge of G_S having a plus end (a minus end respectively). Clearly, $S = S_+ \cup S_-$ holds, but it is possible that $S_+ \cap S_- \neq \emptyset$. We call G_S (or simply, S) a biclique in G if:

(i) G_S is completely connected;

(ii) $S_+ \cap S_- = \emptyset$ holds; and

(iii) G_S is a maximal subgraph of G with respect to the requirements (i) and (ii).

For the bigraph

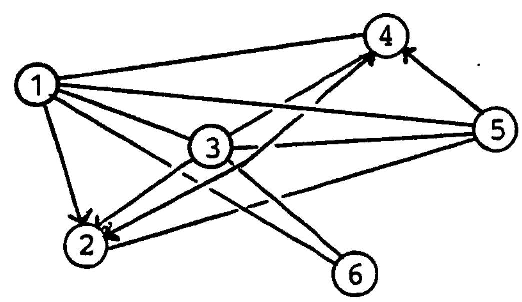

Figure 5

shown in Figure 5, the nodes 1,2,3 form a biclique with $S_+ = \{1,3\}$ and $S_- = \{2\}$. Although nodes 1,2,3,4 form a clique, they do not form a biclique because node 4 is met by both plus ends and minus ends of edges connecting node 4 to nodes 1,2,3. Similarily, nodes 1,2,3,5 form a clique but not a biclique because now node 2 is met by both plus ends and minus ends of edges connecting it to nodes 1,3,5. Nodes 1,2,3,6 do not form a biclique because they are not completely connected. Thus, nodes 1,2,3 are maximal with respect to the required property and do form a biclique.

A biclique of G with node set $S = S_+ \cup S_-$ is called a strong biclique of G if there does exist a node $k \notin S$ with edges $[k,i]$ for all $i \in S$ in G such that the edge $[k,i]$ has a plus end meeting i if $i \in S_+$ and a minus end meeting i if $i \in S_-$. In figure 5, nodes 1,2,3 do not form a strong biclique because of node 4. The strong bicliques in the graph shown in figure 5 are $\{1,3,4\}$, $\{1,3,6\}$, $\{2,4\}$, and $\{2,5\}$.

For a bigraph G and a biclique of G with nodes $S = S_+ \cup S_-$, define the corresponding <u>clique inequality</u> to be the inequality

$$\sum_{j \in S_+} x_j - \sum_{j \in S_-} x_j \geq |S_+| - 1$$

<u>Proposition 6:</u> *Every $0-1$ solution for a bigraph G satisfies the clique inequalities of G .*

<u>Proof:</u> The clique inequality for $S = S_+ \; S_-$ can be written

$$\sum_{j \in S_+} (1 - x_j) + \sum_{j \in S_-} x_j \leq 1 \quad .$$

In order to violate it, there must be one of:

(i) $x_i = 1$ and $x_j = 1$ for $i \in S_-$ and $j \in S_-$;
(ii) $x_i = 1$ and $x_j = 0$ for $i \in S_-$ and $j \in S_+$;
(iii) $x_i = 0$ and $x_j = 0$ for $i \in S_+$ and $j \in S_+$.

Each of these three possibilities is explicitly excluded by an edge of G , as illustrated in Figure 6.

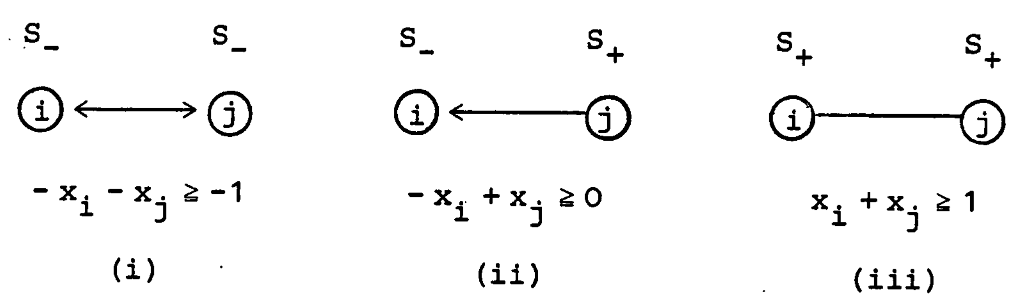

Figure 6

<u>Proposition 7.</u> *For a bigraph G and a biclique with node set $S = S_+ \cup S_-$ of G , if T is a subset of S , then the inequality*

$$\sum_{j \in T_+} x_j - \sum_{j \in T_-} x_j \geq |T_+| - 1 \quad ,$$

where $T_+ = T \cap S_+$ and $T_- = T \cap S_-$, is satisfied by every $0-1$ solution *for* G *. However, this inequality for* T *is implied by the clique inequality corresponding to* S *and the linear inequalities:* $0 \le x_j \le 1$*, for all* $j \in \mathbb{N}$ *.*

Proof. The fact that every $0-1$ solution satisfies this inequality corresponding to T has the same proof as for proposition 6.

The second assertion is shown by adding the inequalities

$$\sum_{j \in S_+} x_j - \sum_{j \in S_-} x_j \ge |S_+| - 1$$

$$-x_j \ge -1 \ , \quad j \in S_+ - T_+ \ ,$$

$$x_j \ge 0 \ , \quad j \in S_- - T_- \ ,$$

to derive the inequality corresponding to T .

Proposition 8. *If a biclique* S *of a bigraph* G *is not a strong biclique, then there exist two bicliques* $C_1 \neq S$ *and* $C_2 \neq S$ *such that the clique constraint corresponding to* S *is implied by the two clique inequalities corresponding to* C_1 *and* C_2*. Thus, the clique constraint for* S *is implied by the system of strong biclique inequalities.*

Proof. Suppose the biclique S with nodes $S = S_+ \cup S_-$ is not strong, i.e. there exists $k \notin S$ such that the edge $[k,i]$ is in G for every $i \in S$ and has a plus end (a minus end, respectively) at the node i when $i \in S_+$ ($i \in S_-$). By the maximality of S, the node k must have some plus ends and some minus ends meeting k for edges $[k,i]$, $i \in S$. Define

$T_+ = \{i \in S \mid \text{edge } [k,i] \text{ has a plus end meeting } k\}$

$T_- = \{i \in S \mid \text{edge } [k,i] \text{ has a minus end meeting } k\}$.

It is clear that $C_1 = \{k\} \cup T_+$ and $C_2 = \{k\} \cup T_-$ are bicliques with

$$C_1 = [\{k\} \cup (S_+ \cap T_+)] \cup [S_- \cap T_+] = C_1^+ \cup C_1^- \ ,$$
$$C_2 = [S_+ \cap T_-] \cup [\{k\} \cup (S_- \cap T_-)] = C_2^+ \cup C_2^-$$

being partitions of C_1 and C_2 so that their clique inequalities are

$$\sum_{j \in C_1^+} x_j - \sum_{j \in C_1^-} x_j \ge |C_1^+| - 1 \ , \text{ and}$$

$$\sum_{j \in C_2^+} x_j - \sum_{j \in C_1^+} x_j \ge |C_2^+| - 1 \ .$$

Adding these two inequalities gives

$$\sum_{j\in S_+} x_j - \sum_{j\in S_-} x_j \geq 1 + |S_+| - 2 = |S_+| - 1,$$

because x_k appears with $a + 1$ and $a - 1$ in the two inequalities. Hence, the first assertion is proven.

Let P denote the polytope defined by all clique constraints and the constraints $0 \leq x_j \leq 1$, $j \in \mathbb{N}$. By theorem 5, P has dimension n because P contains the convex hull of $0 - 1$ solutions. Consequently, every facet of P has dimension $n-1$. Let $ax \geq a_o$ be an inequality corresponding to a biclique which is not strong. By the first part of the proof, there exists two clique inequalities $cx \geq c_o$ and $dx \geq d_o$ such that $a = c + d$ and $a_o = c_o + d_o$. Hence, $ax = a_o$ if and only if $cx = c_o$ and $dx = d_o$. Furthermore, the $2 \times n$ matrix with rows c and d has rank 2. It follows that $ax = a_o$ defines a face of P of dimension at most $n-2$. The second assertion of the proposition is, thus, proven.

Proposition 6,7, and 8 prove the following result.

Theorem 9. *For a bigraph G the set of $0-1$ solutions to the inequality system*

$$0 \leq x_j \leq 1 \quad , \quad j \in \mathbb{N}$$

(4.1) $$\sum_{j\in T_+} x_j - \sum_{j\in T_-} x_j \geq |T_+| - 1 \quad \text{for all } T \subseteq C \text{ and for all } C\text{, } C \text{ a biclique in } G\text{.}$$

are exactly the $0-1$ solutions for G. Furthermore, the inequality system

$$0 \leq x_j \leq 1 \quad , \quad j \in \mathbb{N}$$

(4.2) $$\sum_{j\in C_+} x_j - \sum_{j\in C_-} x_j \geq |C_+| - 1 \quad \text{for all strong bicliques } C \text{ in } G\text{.}$$

has the same (real) solutions x $\mathbb{R}^n$ as the (, in general, considerably larger) system (4.1).

Proof. By Proposition 6 and 7 we have that every $0-1$ solution for G satisfies the system (4.1). On the other hand, since every edge of G is contained in some biclique of G it follows by the second part of Proposition 7 that every $0-1$ solution to (4.1) is a $0-1$ solution for G. Again by Proposition 7 it follows that in the inequality system (4.1) we can restrict ourselves to considering bicliques only without changing the real solution space. By Proposition 8, if a biclique C is not a strong biclique, then the resulting inequality is implied by the clique inequalities associated with the strong bicliques. Thus Theorem 9 follows.

When the bigraph G has only $(-,-)$ edges the original inequalities are all of the type $-x_i - x_j \geq -1$, the strong bicliques are cliques in the usual meaning (and vice versa), and the form of the clique inequality system is of the form related to graphs. With respect to graphs it is known [5] that all clique inequalities define facets of the associated $0-1$ polytope.

For general bigraphs, we can ask the question whether any strong biclique inequality can be omitted without changing the solution set in $\mathbb{R}^n$. The next theorem answers the question in the negative.

Theorem 10. *The strong biclique inequalities are facets of the convex hull of $0-1$ solutions for the bigraph G .*

Proof. The proof follows the lines of the proof for undirected graphs [10] but is more difficult.

We must show, by Theorem 5, that there exist n affinely independent $0-1$ solution vectors satisfying the strong biclique inequality with equality. Let $C = C_+ \cup C_-$ be the node set of a strong biclique and let $c_1 = |C_+|$, $c_2 = |C_-|$ and $c = |C|$. Assume without loss of generality that $C = \{1,\ldots,c\}$. Construct c_1 affinely independent $0-1$ solutions for G by setting $x_i = 0$ for exactly one $i \in C_+$ at a time using Construction 0. By Proposition 3 such $0-1$ solution exists. Construct c_2 affinely independent $0-1$ solutions for G by setting $x_i = 1$ for exactly one $i \in C_-$ at a time using Construction 1 and invoking Proposition 3. The resulting c $0-1$ solutions for G satisfy the clique inequality with equality and are affinely independent among themselves as follows by construction the $c \times c$ matrix on the first c columns corresponding to the nodes in C .

To complete the proof we construct $2(n-c)$ $0-1$ solutions for G satisfying the inequality with equality as follows: Since C is a strong biclique for each node $k \notin C$ there exist at least one node $i \in C$ such that either $i \in C_+$ and there is no edge $[i,k]$ with a plus end at i , or $i \in C_-$ and there is no edge $[i,k]$ with a minus end at node i . In the former case, set $x_i = 0$ and use Construction 0; in the latter case, set $x_i = 1$ and use Construction 1. In either case, the construction fixes all x_j with $j \in C$ so that the clique inequality is satisfied with equality and so that the node k satisfies $k \in \mathbb{R}$ in Construction 0 or $k \in S$ in Constuction 1 .

By Proposition 1, the resulting subgraph G_R (G_S , respectively) is a bigraph and we can apply Proposition 3 to G_R (G_S respectively). Combining the statements it follows that for each $k \notin C$ there exist two solutions x^o and x^1 each satisfying the clique inequality with equality and such that $x_j^o = x_j^1$ for all $j \neq k$, $x_k^o = 0$ and $x_k^1 = 1$ hold. We list the $2(n-c)$ 0-1 solutions for G thus obtained pairwise and observe that by elementary row operation we obtain a $(n-c) \times (n-c)$ identity matrix in columns $c+1,\ldots,n$. Consequently, by construction, we have n affinely independent 0-1 solutions to G satisfying the clique inequality with equality and thus Theorem 10 follows.

We note that - like in the case of set-packing polyhedra, see [10],[11],[13] - it is natural to look for other facet-defining structures in a bigraph G - generalizing the notion of cdd cycles, webs, etc. Another avenue for research is a generalization of the "lifting procedure" for facets to the case of bigraphs. These possibilities are at present left for future research.

5. Biperfect Graphs

In this section, we define the notion of biperfect graphs, give some examples and simple results, and make several conjectures. The principal conjecture is that the class of biperfect graphs is really the same as the class of perfect graphs.

For a given bigraph G define its clique matrix M to be the matrix whose rows consist of the coefficients of the strong biclique inequalities. Define the clique polytope of G to be the set of $x \in \mathbb{R}^n$ satisfying $0 \leq x \leq 1$ and satisfying the clique inequalities for strong bicliques of G. We are interested in the question of when every vertex of the clique polytope is a 0-1 vector.

Another way to view the clique inequalities is to start with an inequality system of the form:

$$0 \leq x_j \leq 1 \ , \ j = 1,\ldots,n \ ;$$
$$\sum_{j=1}^{n} a_{ij}x_j \geq p_i - 1 \ , \ i = 1,\ldots,m \ ;$$

where every a_{ij} is 0, 1 or -1 and p_i is equal to the number of +1's among $a_{i1},\ldots,a_{in}$. When all a_{ij} are 0 and -1, we have a set packing problem, and we wish to generalize the clique approach and perfect graph framework [5] for that problem. Given an inequality system with 0, 1, -1 coefficients we can form a bidirected graph G by making a biclique among the nodes $N = \{1,2,\ldots,n\}$ for each row of $A = (a_{ij})$. That is, put in G an edge $[j,k]$ of type (a_{ij}, a_{ik}) whenever there is a row i with $a_{ij} \neq 0$ and $a_{ik} \neq 0$. Then the 0-1 solutions for this bidirected graph G are the same as the 0-1 solutions for the original system. The particular right-hand side, $p_i - 1$, required in the original system is critical here. The original inequality system cannot have a polytope with only 0-1 vertices unless every strong biclique inequality is among the rows of A, that is, unless the clique matrix M of G is included as a submatrix of A. In general, that inclusion is not sufficient to assure 0-1 verices, but the clique matrix M does give a smaller polytope of solutions, or a tighter linear programming relaxation than the original system of linear inequalities.

We could thus define biperfect graphs in terms of polytopes of 0-1 vertices, but we instead follow the classical approach for perfect graphs [5].

First, let us discuss what it means to bidirect an undirected graph G^o. An undirected graph has no sign, + or - , assigned to the ends of edges. To bidirect G^o means to assign a plus or minus to each end of each edge, giving edges which may be of type (+,+),(+,-),(-,+) or (-,-). To bidirect G^o to form a bigraph G means to bidirect G^o to give a bigraph G, that is, to give a bidirected graph which is simple (which is sure if G^o is simple) and which is equal to its transitive closure. The latter condition is rather stringent since, for example, we can only bidirect a grah G^o to a bigraph G having only (+,-) or (-,+) edges if G^o is a comparability graph [6]. Any undirected graph G^o can, however, always be given some bidirection just by making every end a plus end or by making every end a minus end. Then, the resulting bidirecged graph G is a bigraph provided only that G^o was simple. We allow other bidirections, one could say, in between these two extreme cases.

Given a bigraph G , we consider optimization problems of the form

$$x_j = 0 \text{ or } 1, \ j = 1,\dots,n \ ;$$

$$\sum_{j\in S_+} x_j - \sum_{j\in S_-} x_j \geq |S_+| - 1 \ , \text{ all strong bicliques } S = S_+ \cup S_- \ ;$$

$$\sum_{j\in Z_+} x_j - \sum_{j\in Z_-} x_j = z \text{ (minimize)},$$

where Z_+ and Z_- are disjoint subsets of N. In other words, we consider optimizing 0-1 solutions for G with objective functions $z = \sum c_i x_j$ having $c_i = 0$, +1 , or -1. Then,

$$Z_+ = \{j \mid c_j = +1\},$$
$$Z_- = \{j \mid c_j = -1\},$$
$$Z_o = \{j \mid c_j = 0\},$$

and Z_o, Z_+, Z_-, form a partition of the nodes.

The linear programming relaxation replaces $x_j = 0$ or 1 by $0 \leq x_j \leq 1$. We form the dual linear program and consider the optimization problem of finding integer answers to it. This dual problem has a variable π_S for each strong biclique and a variable δ_j for each node. The constraints

$$\pi_S \geq 0 \text{ and integer for all strong bicliques ;}$$
$$\delta_j \geq 0 \ , \ j = 1,\dots,n \ ;$$

$$\sum_{S:j\in S_+} \pi_S - \sum_{S:j\in S_-} \pi_S - \delta_j \leq \begin{cases} 0, & j \in Z_o \\ 1, & j \in Z_+ \\ -1, & j \in Z_- \end{cases}$$

$$\sum_S (|S_+| - 1)\pi_S - \sum_{j=1}^{n} \delta_j = v \text{ (maximize).}$$

The summations over S are sums over all strong bicliques $S = S_+ \cup S_-$. In general, the objective v of the dual problem satisfies $v \leq z$ for z the objective fo the original problem for G. We say a bigraph G has the (strong) max-min property if these two objectives are equal, vide Fulkerson [5 ,p.172].

Define an undirected graph G^o to be biperfect if every bigraph G formed by bidirecting G^o satisfies the max-min property.

A biperfect graph is obviously perfect because one way of directing an undirected graph is to make all ends be minus ends, and then the max-min property becomes the famous relation (maximum number of independent nodes) = = (min number of cliques needed to cover all nodes)[2].

We conjecture that a graph is biperfect if and only if it is perfect. Some reasons for thinking that the conjecture may be true will be given.

Proposition 11 *Bipartite graphs are biperfect.*

Proof. A bipartite graph has no triangles so has no cliques other than the edges themself. For a bipartite graph G^o, the dual linear program for some bidirection of G^o is:

$$\pi_e \geq 0 \text{ and integer, all edges } e \text{ ;}$$
$$\delta_i \geq 0 \text{ , all nodes } i \text{ ;}$$

$$\sum_{\substack{e \text{ meets } i \\ \text{with a + end}}} \pi_e - \sum_{\substack{e \text{ meets } i \\ \text{with a - end}}} \pi_e - \delta_i \leq \begin{cases} 0 \text{ , } i \in Z_o \\ 1 \text{ , } i \in Z_+ \\ -1 \text{ , } i \in Z_- \end{cases}$$

$$\sum_{\substack{e \text{ has two} \\ + \text{ ends}}} \pi_e - \sum_{\substack{e \text{ has two} \\ - \text{ ends}}} \pi_e - \sum_{i=1}^{n} \delta_i = v \text{ (maximize)}$$

A linear program of this type has integer answers to both its primal and dual whenever there does not exist any odd circuit [4], where here an odd circuit means a circuit with an odd number of edges having either two plus ends or two minus ends. We can obviously bidirect a bipartite graph with a circuit to have an odd circuit just by making all ends plus ends except one (see Figure 7).

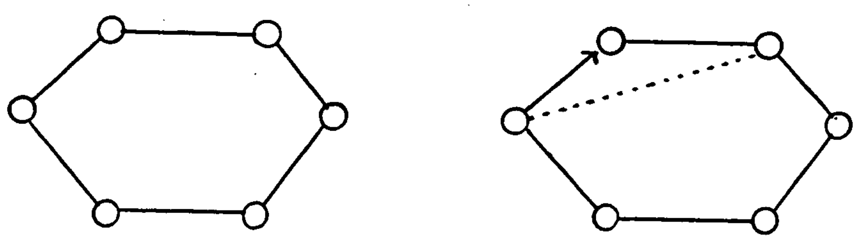

Figure 7

However, doing so causes a triangle to be formed in the transitive closure so that the resulting bidirected grah could not be transitively closed since we started with a gipartite graph, which has no triangles.

In general, any bidirection of a bipartite graph in order to be transitively closed will have to make both ends meeting a node in any circuit either both plus er both minus ends. Then, the resulting circuit will not be an odd circuit as can be seen by a simple parity argument. Therefore, any bipartite graph is biperfect.

This proof suggest the next proposition. First, define an odd-dihole to be a hole (i.e. a circuit with no chords) having an odd number of edges with either both plus ends or both minus ends. An odd hole in an undirected graph is a hole with an odd number of edges. Any bidirected graph with an odd-dihole does not satisfy the max-min property.

<u>Proposition 12</u> *An undirected graph* G^o *with an odd hole has an odd-dihole in every bigraph* G *formed by bidirecting* G^o *. Conversely, if a bigraph* G *has an odd-dihole, then the undirected graph* G^o *formed by just dropping the* $+$ *or* $-$ *sign on each end of edges has an odd hole.*

The proof is essentially that already given in proving proposition 11. What proposition 12 says is that if a graph is not biperfect because some bigraph formed from it had an odd-dihole, then the original graph was not perfect because it had an odd hole. However, it says something in the other direction as well; namely, if a graph is not perfect because it has an odd hole, then every bigraph formed from it will not have the max-min property because of an odd-dihole.

We conjecture a converse of the previous conjecture: if some bigraph G formed from G^o has the max-min property, then G^o is biperfect. That is, all bigraphs formed from G^o have the max-min property or none do.

An interesting example is given by comparability graphs. A comparability graph is an undirected graph such that directions can be assigned to each edge so that the resutling directed graph is acyclic and transitively closed. The ordering then given by $i > j$ if an edge has a plus end meeting i and a minus end meeting j is a partial order. Such graphs (the original undirected versions) are known to be perfect. The strong max-min property for such graphs says that the maximal number of pair-wise incomparable elements in a partial order is equal to the minimum number of chains (linearly ordered subsets) covering all elements. Although a comparability graph may have several partial orders which can be formed by directing the edges, the incomparable elements and the chains are the same in all such partial orders.

We do not know if comparability graphs are biperfect. We conjecture that they are. However, one way of bidirecting the edges, other than making all plus ends, that works is to direct the edges so as to give a partial order. The max-min property holds because the resulting matrix is totally unimodular, being a network flow matrix. The max-min property has an interesting statement here. First, define a node with a $+1$ cost coefficient to be a <u>source</u> and a node with a -1 cost coefficient to be a <u>sink</u>. Define an <u>upper dominated set</u> to be a set S of nodes such that if $j \in S$ and $i > j$ then $i \in S$. The primal problem is to find an upper dominated set with the largest surplus of sinks minus sources. The dual problem amounts to finding pairings of sources to sinks by directed edges so that each source is paired to only one sink and the fewest number of sinks are left unpaired. The max-min property is easy to show from linear programming duality and total unimodularity of the coefficient matrix.

If perfect graphs are perfect it suffices to show the max-min property for the directed version of the problem. The reason that this could not be used for graphs other than comparability graphs is that only comparability graphs gave a directed version which is simple and transitively closed.

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